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Elliptically contoured random fields in space and time

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Abstract

Elliptically contoured random fields are natural extensions of the Gaussian random field, and may or may not have second-order moments. A second-order elliptically contoured random field is determined by its mean and covariance functions, just like the Gaussian one. This paper proposes a class of covariance functions for second-order elliptically contoured random fields by using the nonnegative mixture method, which possess the following properties: nonseparable, nonstationary but with the stationary case as a special case, and allowing for both positive and negative correlation. Some long-range dependent models are also derived.

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1. Introduction

To analyze and model spacetime uncertainty in various geophysical, informational, environmental, biological and economic systems, we often treat the data as the realizations of spacetime random fields and then employ statistical and probabilistic techniques to describe observed variabilities, to model the data, and to predict future or neighborhood values [3, 7, 12, 18, 19, 21, 23–25, 32–34, 37]. One of the most popularly used random fields is the Gaussian one, which is completely characterized by its mean and covariance functions. In practice, however, there are often specific reasons for assuming particular non-Gaussian finite-dimensional distributions. For instance, the study of Euclidean quantum field theory in physics is essentially the study of Gaussian and related random fields, but it is really only the non-Gaussian ones that are physically interesting [1]. While sea surfaces are often modeled as random fields, a Gaussian field is not a precise model of reality, a fact confirmed by the non-Gaussian nature of some of the empirical statistics complied from sea surface data [31]. Non-Gaussian processes arise naturally in statistical physics, and occur typically in situations where coalescence may occur between entities (domain, cluster, etc) of comparable size [8]. Several quantities arising in practical engineering problems (e.g. material, geometric properties, soil properties, wind, wave, earthquake loads) exhibit non-Gaussian probabilistic characteristics [13, 14, 15, 38]. The signals and noise encountered in signal processing environment are often not Gaussian (e.g. [36]). This motivates us to construct non-Gaussian random fields with various properties for theoretical study and practical use.

Suppose that $\{Z(\mathbf{s}; t), \mathbf{s} \in \mathcal{S}, t \in \mathcal{T}\}$ is a real-valued random field over a spacetime index domain $\mathcal{S} \times \mathcal{T}$, where $\mathcal{S}$ equals $\mathbb{R}^d$ or $\mathbb{Z}^d$ and $\mathcal{T}$ equals $\mathbb{R}$ or $\mathbb{Z}$. The variogram (or structure function) and covariance function of the random field are respectively defined by

$$\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2) = \tfrac{1}{2}\operatorname{var}(Z(\mathbf{s}_1; t_1) - Z(\mathbf{s}_2; t_2)),$$

and

$$C(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2) = \mathrm{E}[\{Z(\mathbf{s}_1; t_1) - \mathrm{E}Z(\mathbf{s}_1; t_1)\}\{Z(\mathbf{s}_2; t_2) - \mathrm{E}Z(\mathbf{s}_2; t_2)\}],$$
$$(\mathbf{s}_k; t_k) \in \mathcal{S} \times \mathcal{T}, \qquad k = 1, 2,$$

whenever they exist, where E denotes the expectation operator and $\operatorname{var}(Z)$ is the variance of a random variable Z. The existence of the covariance function implies that of the variogram, with the relationship

$$\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2) = \tfrac{1}{2}C(\mathbf{s}_1, \mathbf{s}_1; t_1, t_1) + \tfrac{1}{2}C(\mathbf{s}_2, \mathbf{s}_2; t_2, t_2) - C(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2),$$
$$(\mathbf{s}_k; t_k) \in \mathcal{S} \times \mathcal{T}, \qquad k = 1, 2,$$

but not vice versa.

For spatio-temporal data analysis in practice, simplifying assumptions such as intrinsic stationarity or (weak, second-order) stationarity are typically required. Here 'stationarity' is often replaced by 'homogeneity'. A random field $\{Z(\mathbf{s}; t), \mathbf{s} \in \mathcal{S}, t \in \mathcal{T}\}$ is said to be stationary in spacetime, if its mean function $\mathrm{E}Z(\mathbf{s}; t)$ is a constant, and its covariance function $C(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2)$ depends only on the spatial lag $\mathbf{s}_1 - \mathbf{s}_2$ and the temporal lag $t_1 - t_2$. In such a case, we write $C(\mathbf{s}_1 - \mathbf{s}_2; t_1 - t_2)$ instead of $C(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2)$ and call it a stationary covariance function on $\mathcal{S} \times \mathcal{T}$.

The intrinsic stationarity is based on the variogram but not on the covariance, and is thus weaker than stationarity. The random field $\{Z(\mathbf{s}; t), \mathbf{s} \in \mathcal{S}, t \in \mathcal{T}\}$ is said to be intrinsically stationary in spacetime (or have stationary increments in spacetime) if, for any $(\mathbf{s}_0; t_0) \in \mathcal{S} \times \mathcal{T}$, the increment

$$\{Z(\mathbf{s} + \mathbf{s}_0; t + t_0) - Z(\mathbf{s}_0; t_0), \mathbf{s} \in \mathcal{S}, t \in \mathcal{T}\}$$

is stationary in spacetime or, equivalently, if $\mathrm{E}\{Z(\mathbf{s} + \mathbf{s}_0; t + t_0) - Z(\mathbf{s}_0; t_0)\}$ is a constant for any $(\mathbf{s}_0; t_0) \in \mathcal{S} \times \mathcal{T}$, and $\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2)$ depends only on $\mathbf{s}_1 - \mathbf{s}_2$ and $t_1 - t_2$. Alternatively, $\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2)$ will be written as $\gamma(\mathbf{s}_1 - \mathbf{s}_2; t_1 - t_2)$ and called an intrinsically stationary variogram on $\mathcal{S} \times \mathcal{T}$.

According to Kolmogorov's existence theorem, a positive definite function on $\mathcal{S} \times \mathcal{T}$ can always be thought of as the covariance function of a zero-mean Gaussian random field. The question is: for a given positive definite function, is there any other possibility for the underlying finite-dimensional distributions? There are rarely few answers to this question, except for the counterexamples in [2] and [29]. Surprisingly, the so-called Gaussian covariance, $\exp(-\|\mathbf{s}\|^2), \mathbf{s} \in \mathbb{R}^d$, cannot be a covariance function associated with any log-Gaussian random field, where $\|\mathbf{s}\| = \left(\sum_{k=1}^d s_k^2\right)^{1/2}$ is the Euclidean norm of $\mathbf{s} \in \mathbb{R}^d$. Also, it is not clear what kind of positive definite function could be the covariance function of a binary random field. Recently, a novel and simple method has been provided in [26] for constructing many non-Gaussian random fields with any given correlation structure and with the corresponding finite-dimensional distributions identified, of which a particular class is elliptically contoured random fields. An elliptically contoured stochastic process is also called a spherically invariant random process, and is defined in [41]. Theorem 2 (iii) of [16] characterizes the spherically invariant process as a scale mixture of the Gaussian process.

Our special attention in this paper is paid to spacetime Gaussian random fields and second-order elliptically contoured random fields, for which the covariance function is the key component. In order to describe a wide range of spacetime dependence and interaction, it is better to have covariance functions taking both positive and negative values. A simple reason for this is that not every mechanism in reality is always nonnegatively correlated in spacetime. Observed covariance functions in practice, especially the ones of climatological background errors, often change sign [40]. To get a more theoretical reason, let us consider a zero-mean Gaussian random field $\{Z(\mathbf{s}; t), \mathbf{s} \in \mathcal{S}, t \in \mathcal{T}\}$ with covariance $C(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2)$. It is known [17, 35, 39] that $C(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2)$ is nonnegative on $\mathcal{S} \times \mathcal{T}$ if and only if, for every positive integer n and any $(\mathbf{s}_k; t_k) \in \mathcal{S} \times \mathcal{T}$ $(k = 1, \ldots, n)$, the random vector $(Z(\mathbf{s}_1; t_1), \ldots, Z(\mathbf{s}_n; t_n))$ is associated [10] in the sense that the inequality

$$\operatorname{cov}(f(Z(\mathbf{s}_1; t_1), \ldots, Z(\mathbf{s}_n; t_n)), g(Z(\mathbf{s}_1; t_1), \ldots, Z(\mathbf{s}_n; t_n))) \geqslant 0$$

holds for each pair of bounded Borel measurable functions f and g that are nondecreasing in each argument. One of the approaches to get a covariance model taking both positive and negative values is to work on linear combination of covariance functions [23–25].

The aim of this paper is to present a class of spacetime Gaussian and second-order elliptically contoured random fields with nonseparable covariance functions that allow for both positive and negative correlations. The general form is nonstationary, but, unlike some previous papers in the literature, contains the stationary case as a special case. Some models with long-range dependence are also derived. The main results are presented in section 2 using the nonnegative mixture approach [25], while section 3 illustrates some parametric examples. Since our covariance functions are semiparametric, section 4 discusses how the parameters affect the model. The proofs of theorems occupy section 5. Section 6 offers a brief discussion on possible applications of the presented class of the random fields in physics and applied sciences.

2. General formulation

In this section we derive a class of spatio-temporal covariance functions. Before presenting our models, let us recall the definition of the completely monotone function. A nonnegative and continuous function $\ell(x)$, $x \geqslant 0$, is completely monotone on $[0, \infty)$, if it has derivatives of all orders on $(0, \infty)$ and the derivatives alternate in sign, i.e. for every positive integer n, $(-1)^n \frac{\mathrm{d}^n}{\mathrm{d}x^n} \ell(x) \geqslant 0, x > 0$. Bernstein's theorem asserts that $\ell(x)$ is completely monotone on $[0, \infty)$ if and only if $\ell(x) = \int_0^\infty \exp(-xu) \mathrm{d}\mu(u)$, where $\mu(u)$ is a nonnegative finite measure on $[0, \infty)$. A recent expository survey of properties of completely monotone functions as well as various examples can be found in [30].

Our main model is formulated in the following theorem.

Theorem 1. *Assume that $\ell(x)$ is a completely monotone function on $[0, \infty)$, α_1 and α_2 are positive constants with $\alpha_1 < \alpha_2$ and $\boldsymbol{\theta}_0 \in \mathbb{R}^d$ is a constant vector. If $\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2)$ is a variogram on $\mathbb{R}^d \times \mathcal{T}$, then*

$$\begin{aligned} C(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2) = &\{\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2) + \alpha_1\}^{-\frac{d}{2}} \ell\left(\frac{\|\mathbf{s}_2 - \mathbf{s}_1 + \boldsymbol{\theta}_0(t_2 - t_1)\|^2}{\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2) + \alpha_1}\right) \\ &- \{\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2) + \alpha_2\}^{-\frac{d}{2}} \ell\left(\frac{\|\mathbf{s}_2 - \mathbf{s}_1 + \boldsymbol{\theta}_0(t_2 - t_1)\|^2}{\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2) + \alpha_2}\right), \end{aligned} \tag{2.1}$$

$(\mathbf{s}_k; t_k) \in \mathbb{R}^d \times \mathcal{T}, \qquad k = 1, 2,$

is the covariance function of a second-order elliptically contoured random field on $\mathbb{R}^d \times \mathcal{T}$.

The covariance model (2.1) is not stationary in space or in time, unless $\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2)$ is intrinsically stationary in space or in time. A simple example of variograms is the separable one

$$\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2) = \gamma_S(\mathbf{s}_1, \mathbf{s}_2) + \gamma_T(t_1, t_2), \qquad \mathbf{s}_1, \mathbf{s}_2 \in \mathbb{R}^d, \qquad t_1, t_2 \in \mathcal{T},$$

where $\gamma_S(\mathbf{s}_1, \mathbf{s}_2)$ and $\gamma_T(t_1, t_2)$ are purely spatial and purely temporal variograms on $\mathbb{R}^d$ and $\mathcal{T}$, respectively. Two other examples are

(i) $\gamma(\mathbf{s}_1 + \mathbf{s}_2; t_1 + t_2) + \gamma(\mathbf{s}_1 - \mathbf{s}_2; t_1 - t_2) - \frac{1}{2}\{\gamma(2\mathbf{s}_1; 2t_1) + \gamma(2\mathbf{s}_2; 2t_2)\}$,
(ii) $\frac{1}{2}\{\gamma(2\mathbf{s}_1; 2t_1) + \gamma(2\mathbf{s}_2; 2t_2)\} - \gamma(\mathbf{s}_1 + \mathbf{s}_2; t_1 + t_2) + \gamma(\mathbf{s}_1 - \mathbf{s}_2; t_1 - t_2)$,

where $\gamma(\mathbf{s}; t)$ is an intrinsically stationary variogram on $\mathbb{R}^d \times \mathcal{T}$, [22].

To have a spacetime covariance that is stationary in space or in time, it requires the corresponding intrinsically stationary assumption for $\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2)$ in the construction (2.1). For instance, if $\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2)$ is intrinsically stationary in space or in time, then (2.1) is stationary in space or in time. As an example,

$$\int_0^{t_1} (t_1 - |u|) C_0(\mathbf{0}; u)\, \mathrm{d}u + \int_0^{t_2} (t_2 - |u|) C_0(\mathbf{0}; u)\, \mathrm{d}u - \int_0^{t_1}\int_0^{t_2} C_0(\mathbf{s}; u - v)\, \mathrm{d}u\, \mathrm{d}v,$$
$$\mathbf{s} \in \mathbb{R}^d, \qquad t_1, t_2 \in \mathbb{R},$$

is a variogram on $\mathbb{R}^d \times \mathbb{R}$ intrinsically stationary in space [24], where $C_0(\mathbf{s}; t)$ is a stationary covariance function on $\mathbb{R}^d \times \mathbb{R}$.

The next corollary follows by assuming that $\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2)$ is intrinsically stationary in spacetime.

Corollary 1. *If $\gamma(\mathbf{s}; t)$ is an intrinsically stationary variogram on $\mathbb{R}^d \times \mathcal{T}$, then*

$$C(\mathbf{s}; t) = \{\gamma(\mathbf{s}; t) + \alpha_1\}^{-\frac{d}{2}} \ell\left(\frac{\|\mathbf{s} + \boldsymbol{\theta}_0 t\|^2}{\gamma(\mathbf{s}; t) + \alpha_1}\right) - \{\gamma(\mathbf{s}; t) + \alpha_2\}^{-\frac{d}{2}} \ell\left(\frac{\|\mathbf{s} + \boldsymbol{\theta}_0 t\|^2}{\gamma(\mathbf{s}; t) + \alpha_2}\right), \qquad (2.2)$$
$$\mathbf{s} \in \mathbb{R}^d, \qquad t \in \mathcal{T},$$

is a stationary covariance function on $\mathbb{R}^d \times \mathcal{T}$.

In particular, when the variogram $\gamma(\mathbf{s}; t)$ in (2.2) is separable, $\gamma(\mathbf{s}; t) = \gamma_S(\mathbf{s}) + \gamma_T(t)$, say, we obtain

$$C(\mathbf{s}; t) = \{\gamma_S(\mathbf{s}) + \gamma_T(t) + \alpha_1\}^{-\frac{d}{2}} \ell\left(\frac{\|\mathbf{s} + \boldsymbol{\theta}_0 t\|^2}{\gamma_S(\mathbf{s}) + \gamma_T(t) + \alpha_1}\right)$$
$$- \{\gamma_S(\mathbf{s}) + \gamma_T(t) + \alpha_2\}^{-\frac{d}{2}} \ell\left(\frac{\|\mathbf{s} + \boldsymbol{\theta}_0 t\|^2}{\gamma_S(\mathbf{s}) + \gamma_T(t) + \alpha_2}\right), \qquad \mathbf{s} \in \mathbb{R}^d, \qquad t \in \mathcal{T},$$

which is stationary on $\mathbb{R}^d \times \mathcal{T}$.

Letting α_2 in (2.1) tend to infinity leads to the following corollary.

Corollary 2. *For a positive constant α,*

$$C(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2) = \{\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2) + \alpha\}^{-\frac{d}{2}} \ell\left(\frac{\|\mathbf{s}_2 - \mathbf{s}_1 + \boldsymbol{\theta}_0(t_2 - t_1)\|^2}{\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2) + \alpha}\right), \qquad (2.3)$$
$$\mathbf{s}_1, \mathbf{s}_2 \in \mathbb{R}^d, \qquad t_1, t_2 \in \mathcal{T},$$

is a covariance function on $\mathbb{R}^d \times \mathcal{T}$.

Many particular cases of (2.3) can be obtained once $\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2)$ is specified. For instance, when $\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2)$ reduces to a purely temporal variogram on $\mathcal{T}$, the model (3.3) of [22] is obtained. It is now clear that (2.1) is essentially the difference of two covariance functions on $\mathbb{R}^d \times \mathcal{T}$. Linear combinations of type (2.3) are presented in the following corollary, in the format analogous to those in [23, 24].

Corollary 3.

(i) If θ is a nonnegative constant, then

$$C(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2) = \theta\{\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2) + \alpha_1\}^{-\frac{d}{2}} \ell\left(\frac{\|\mathbf{s}_2 - \mathbf{s}_1 + \boldsymbol{\theta}_0(t_2 - t_1)\|^2}{\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2) + \alpha_1}\right) + (1-\theta)\{\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2) + \alpha_2\}^{-\frac{d}{2}} \ell\left(\frac{\|\mathbf{s}_2 - \mathbf{s}_1 + \boldsymbol{\theta}_0(t_2 - t_1)\|^2}{\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2) + \alpha_2}\right), \tag{2.4}$$

$$\mathbf{s}_1, \mathbf{s}_2 \in \mathbb{R}^d, \qquad t_1, t_2 \in \mathcal{T},$$

is a covariance function on $\mathbb{R}^d \times \mathcal{T}$.

(ii) Conversely, if (2.4) is a covariance function on $\mathbb{R}^d \times \mathcal{T}$ in all dimensions $\mathbb{R}^d$, and $\ell(0) > 0$, then the constant θ has to be nonnegative.

One proof of the validity of (2.4) is by the observation that it is the sum of two covariances

$$\theta\left[\{\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2) + \alpha_1\}^{-\frac{d}{2}} \ell\left(\frac{\|\mathbf{s}_2 - \mathbf{s}_1 + \boldsymbol{\theta}_0(t_2 - t_1)\|^2}{\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2) + \alpha_1}\right) - \{\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2) + \alpha_2\}^{-\frac{d}{2}} \ell\left(\frac{\|\mathbf{s}_2 - \mathbf{s}_1 + \boldsymbol{\theta}_0(t_2 - t_1)\|^2}{\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2) + \alpha_2}\right)\right] + \{\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2) + \alpha_2\}^{-\frac{d}{2}} \ell\left(\frac{\|\mathbf{s}_2 - \mathbf{s}_1 + \boldsymbol{\theta}_0(t_2 - t_1)\|^2}{\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2) + \alpha_2}\right),$$

$$\mathbf{s}_1, \mathbf{s}_2 \in \mathbb{R}^d, \qquad t_1, t_2 \in \mathcal{T}.$$

On the other hand, if (2.4) is a covariance function on $\mathbb{R}^d \times \mathcal{T}$ in all dimensions $\mathbb{R}^d$, then

$$0 \leqslant C(\mathbf{s}_1, \mathbf{s}_1; t_1, t_1)\alpha_1^{\frac{d}{2}}/\ell(0) = \theta + (1-\theta)(\alpha_2/\alpha_1)^{-\frac{d}{2}},$$

and taking the limit as $d \to \infty$ yields $\theta \geqslant 0$.

Corollary 4. *The function*

$$C(\mathbf{s}; t) = \alpha_2^{\frac{d}{2}} \ell(\alpha_2\|\mathbf{s} + \boldsymbol{\theta}_0 t\|^2) - \alpha_1^{\frac{d}{2}} \ell(\alpha_1\|\mathbf{s} + \boldsymbol{\theta}_0 t\|^2), \qquad \mathbf{s} \in \mathbb{R}^d, \qquad t \in \mathcal{T},$$

is a stationary covariance function on $\mathbb{R}^d \times \mathcal{T}$.

Corollary 4 follows from theorem 1 by taking $\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2) \equiv 0$ and substituting the vector (α_1, α_2) with $(1/\alpha_2, 1/\alpha_1)$. For another format see theorem 1 (ii) of [23].

With more parameters involved, a straightforward extension of (2.1) is

$$C(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2) = \sum_{k=1}^{p} (-1)^{k-1}\{\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2) + \alpha_k\}^{-\frac{d}{2}} \ell\left(\frac{\|\mathbf{s}_2 - \mathbf{s}_1 + \boldsymbol{\theta}_0(t_2 - t_1)\|^2}{\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2) + \alpha_k}\right),$$

$$(\mathbf{s}_k; t_k) \in \mathbb{R}^d \times \mathcal{T}, \qquad k = 1, 2,$$

where p is a positive integer and $0 < \alpha_1 < \cdots < \alpha_p$. Another type of parameterization is included in theorem 4 of section 4.

The model (2.5) in the next theorem may be interpreted as the negative of the partial derivative of (2.3) with respect to α, and its validity follows from theorem 1.

Theorem 2. *Assume that $\ell(x)$ is a completely monotone function on $[0,\infty)$ with a finite derivative on the right-hand side of the origin, α is a positive constant and that $\boldsymbol{\theta}_0 \in \mathbb{R}^d$ is a constant vector. If $\gamma(\mathbf{s}_1,\mathbf{s}_2;t_1,t_2)$ is a variogram on $\mathbb{R}^d \times \mathcal{T}$, then*

$$C(\mathbf{s}_1,\mathbf{s}_2;t_1,t_2) = \{\gamma(\mathbf{s}_1,\mathbf{s}_2;t_1,t_2)+\alpha\}^{-\frac{d+2}{2}} \left\{ \frac{d}{2}\ell\left(\frac{\|\mathbf{s}_2-\mathbf{s}_1+\boldsymbol{\theta}_0(t_2-t_1)\|^2}{\gamma(\mathbf{s}_1,\mathbf{s}_2;t_1,t_2)+\alpha}\right) + \|\mathbf{s}_2-\mathbf{s}_1+\boldsymbol{\theta}_0(t_2-t_1)\|^2\{\gamma(\mathbf{s}_1,\mathbf{s}_2;t_1,t_2)+\alpha\}^{-1}\ell'\left(\frac{\|\mathbf{s}_2-\mathbf{s}_1+\boldsymbol{\theta}_0(t_2-t_1)\|^2}{\gamma(\mathbf{s}_1,\mathbf{s}_2;t_1,t_2)+\alpha}\right)\right\},$$

$$(\mathbf{s}_k;t_k) \in \mathbb{R}^d \times \mathcal{T}, \qquad k=1,2, \tag{2.5}$$

is the covariance function of a second-order elliptically contoured random field on $\mathbb{R}^d \times \mathcal{T}$.

The existence of $\ell'(0)$ is obviously required in theorem 2 since, otherwise, $C(\mathbf{s},\mathbf{s};t,t)$ would not be defined. This excludes some completely monotone functions, such as $\ell(x) = \exp(-x^\nu)$, $x \geqslant 0$, where $\nu \in (0,1)$ is a constant.

Corollary. *If $\gamma(\mathbf{s};t)$ is an intrinsically stationary variogram on $\mathbb{R}^d \times \mathcal{T}$, then*

$$C(\mathbf{s};t) = \{\gamma(\mathbf{s};t)+\alpha\}^{-\frac{d+2}{2}} \left\{\frac{d}{2}\ell\left(\frac{\|\mathbf{s}+\boldsymbol{\theta}_0 t\|^2}{\gamma(\mathbf{s};t)+\alpha}\right) + \ell'\left(\frac{\|\mathbf{s}+\boldsymbol{\theta}_0 t\|^2}{\gamma(\mathbf{s};t)+\alpha}\right)\frac{\|\mathbf{s}+\boldsymbol{\theta}_0 t\|^2}{\gamma(\mathbf{s};t)+\alpha}\right\},$$

$$\mathbf{s} \in \mathbb{R}^d, \qquad t \in \mathcal{T},$$

is a stationary covariance function on $\mathbb{R}^d \times \mathcal{T}$.

Note that $\ell(x) \equiv 1$, $x \geqslant 0$, is a completely monotone function on $[0,\infty)$. In such a case, (2.1) reduces to

$$C(\mathbf{s}_1,\mathbf{s}_2;t_1,t_2) = \{\gamma(\mathbf{s}_1,\mathbf{s}_2;t_1,t_2)+\alpha_1\}^{-\frac{d}{2}} - \{\gamma(\mathbf{s}_1,\mathbf{s}_2;t_1,t_2)+\alpha_2\}^{-\frac{d}{2}},$$

$$\mathbf{s}_1,\mathbf{s}_2 \in \mathbb{R}^d, \qquad t_1,t_2 \in \mathcal{T}.$$

This is actually a special case of the following theorem.

Theorem 3. *Assume that ν, α_1 and α_2 are positive constants with $\alpha_1 < \alpha_2$, and that $\gamma(\mathbf{s}_1,\mathbf{s}_2;t_1,t_2)$ is a variogram on $\mathcal{S} \times \mathcal{T}$.*

(i) The function

$$C(\mathbf{s}_1,\mathbf{s}_2;t_1,t_2) = \{\gamma(\mathbf{s}_1,\mathbf{s}_2;t_1,t_2)+\alpha_1\}^{-\nu} - \{\gamma(\mathbf{s}_1,\mathbf{s}_2;t_1,t_2)+\alpha_2\}^{-\nu},$$
$$(\mathbf{s}_k;t_k) \in \mathcal{S} \times \mathcal{T}, \qquad k=1,2, \tag{2.6}$$

is the covariance function of a second-order elliptically contoured random field on $\mathcal{S} \times \mathcal{T}$.

(ii) If θ is a nonnegative constant, then

$$C(\mathbf{s}_1,\mathbf{s}_2;t_1,t_2) = \theta\{\gamma(\mathbf{s}_1,\mathbf{s}_2;t_1,t_2)+\alpha_1\}^{-\nu} + (1-\theta)\{\gamma(\mathbf{s}_1,\mathbf{s}_2;t_1,t_2)+\alpha_2\}^{-\nu},$$
$$(\mathbf{s}_k;t_k) \in \mathcal{S} \times \mathcal{T}, \qquad k=1,2, \tag{2.7}$$

is the covariance function of a second-order elliptically contoured random field on $\mathcal{S} \times \mathcal{T}$.

(iii) If (2.7) is a covariance function on $\mathcal{S} \times \mathcal{T}$ for all positive ν, then θ must be a nonnegative constant.

Although (2.6) is the difference of two covariances, it is always nonnegative, and so is (2.7). In contrast, the following covariance function changes sign in case ν is greater than $d/2$,

$$C(\mathbf{s}_1,\mathbf{s}_2;t_1,t_2) = \{\gamma(\mathbf{s}_1,\mathbf{s}_2;t_1,t_2)+\alpha_1\}^{-\frac{d}{2}+\nu}\{\|\mathbf{s}_2-\mathbf{s}_1+\boldsymbol{\theta}_0(t_2-t_1)\|^2 + \gamma(\mathbf{s}_1,\mathbf{s}_2;t_1,t_2)+\alpha_1\}^{-\nu} - \{\gamma(\mathbf{s}_1,\mathbf{s}_2;t_1,t_2)+\alpha_2\}^{-\frac{d}{2}+\nu}\{\|\mathbf{s}_2-\mathbf{s}_1 + \boldsymbol{\theta}_0(t_2-t_1)\|^2 + \gamma(\mathbf{s}_1,\mathbf{s}_2;t_1,t_2)+\alpha_2\}^{-\nu}, \qquad \mathbf{s}_1,\mathbf{s}_2 \in \mathbb{R}^d, \qquad t_1,t_2 \in \mathcal{T},$$

which is obtained from (2.1) by choosing $\ell(x) = (1+x)^{-\nu}$, $x \geqslant 0$.

For a variogram $\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2)$ on $\mathcal{S} \times \mathcal{T}$, it is known [22] that

$$|\gamma^{1/2}(\mathbf{s}_1, \mathbf{0}; t_1, 0) - \gamma^{1/2}(\mathbf{s}_2, \mathbf{0}; t_2, 0)| \leqslant \gamma^{1/2}(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2) \leqslant \gamma^{1/2}(\mathbf{s}_1, \mathbf{0}; t_1, 0) + \gamma^{1/2}(\mathbf{s}_2, \mathbf{0}; t_2, 0), \quad \mathbf{s}_1, \mathbf{s}_2 \in \mathcal{S}, \quad t_1, t_2 \in \mathcal{T},$$

which implies that

$$\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2) \leqslant 2\gamma(\mathbf{s}_1, \mathbf{0}; t_1, 0) + 2\gamma(\mathbf{s}_2, \mathbf{0}; t_2, 0), \qquad \mathbf{s}_1, \mathbf{s}_2 \in \mathcal{S}, \qquad t_1, t_2 \in \mathcal{T},$$

and that $\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2)$ increases at most as fast as $a_1\|\mathbf{s}_1|^2 + a_2\|\mathbf{s}_2\|^2$ for some positive constants a_1 and a_2. Therefore, (2.6) and (2.7) have long-range dependence [9].

For example, when $\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2) = a_1\|\mathbf{s}_2 - \mathbf{s}_1\|^{\nu_1} + a_2|t_2 - t_1|^{\nu_2}$, where $a_1, a_2, \nu_1 \in (0, 2]$ and $\nu_2 \in (0, 2]$ are positive constants, (2.6) becomes

$$C(\mathbf{s}; t) = (a_1\|\mathbf{s}_2 - \mathbf{s}_1\|^{\nu_1} + a_2|t_2 - t_1|^{\nu_2} + \alpha_1)^{-\nu} - (a_1\|\mathbf{s}_2 - \mathbf{s}_1\|^{\nu_1} + a_2|t_2 - t_1|^{\nu_2} + \alpha_2)^{-\nu}, \quad \mathbf{s} \in \mathbb{R}^d, \quad t \in \mathbb{R},$$

and decays in power law.

As another example, let

$$\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2) = \ln(1 + a_1\|\mathbf{s}_2 - \mathbf{s}_1\|^{\nu_1} + a_2|t_2 - t_1|^{\nu_2}),$$

where $a_1, a_2, \nu_1 \in (0, 2]$ and $\nu_2 \in (0, 2]$ are positive constants. Then (2.6) decays in the logarithm law with

$$C(\mathbf{s}; t) = \{\ln(1 + a_1\|\mathbf{s}_2 - \mathbf{s}_1\|^{\nu_1} + a_2|t_2 - t_1|^{\nu_2}) + \alpha_1\}^{-\nu} - \{\ln(1 + a_1\|\mathbf{s}_2 - \mathbf{s}_1\|^{\nu_1} + a_2|t_2 - t_1|^{\nu_2}) + \alpha_2\}^{-\nu}, \qquad \mathbf{s} \in \mathbb{R}^d, \qquad t \in \mathbb{R}.$$

3. Parametric examples

Our main model (2.1) gives a much richer class of spacetime covariance models in terms of a wide selection of the completely monotone function $\ell(x)$ and the variogram $\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2)$, which one can specialize to meet practical needs such as the data structure, property or convenience. In this section we give two parametric examples of the model (2.2).

Example 1. Let $\ell(x) = \exp(-x^\nu)$, $x \geqslant 0$, and let $\gamma(\mathbf{s}; t) = \|\mathbf{s}\|^{\nu_1} + |t|^{\nu_2}$, where $\nu \in (0, 1]$, $\nu_1 \in (0, 2]$ and $\nu_2 \in (0, 2]$ are constants. Then (2.2) reads

$$C(\mathbf{s}; t) = \{\|\mathbf{s}\|^{\nu_1} + |t|^{\nu_2} + \alpha_1\}^{-d/2} \exp\left\{-\frac{\|\mathbf{s} + \boldsymbol{\theta}_0 t\|^{2\nu}}{(\|\mathbf{s}\|^{\nu_1} + |t|^{\nu_2} + \alpha_1)^\nu}\right\} - \{\|\mathbf{s}\|^{\nu_1} + |t|^{\nu_2} + \alpha_2\}^{-d/2} \exp\left\{-\frac{\|\mathbf{s} + \boldsymbol{\theta}_0 t\|^{2\nu}}{(\|\mathbf{s}\|^{\nu_1} + |t|^{\nu_2} + \alpha_2)^\nu}\right\}, \quad \mathbf{s} \in \mathbb{R}^d, \quad t \in \mathbb{R}. \tag{3.1}$$

This function decays exponentially. By numerical calculation, one can see that it takes negative values. Also, it is differentiable with respective to both $\mathbf{s}$ and t if, for instance, $\nu = 1$ and $\nu_1 = \nu_2 = 2$. A plot of the correlation function, $C(\mathbf{s}; t)/C(\mathbf{0}; 0)$, is illustrated in figure 1 with particular choice of parameters.

Example 2. In (2.2) choose $\ell(x) = \ln \frac{x+b_2}{x+b_1}$, $x \geqslant 0$, and $\gamma(\mathbf{s}; t) = \|\mathbf{s} - \boldsymbol{\theta}_0 t\|^\nu$ where $0 < b_1 < b_2$, and $0 < \nu \leqslant 2$. We obtain

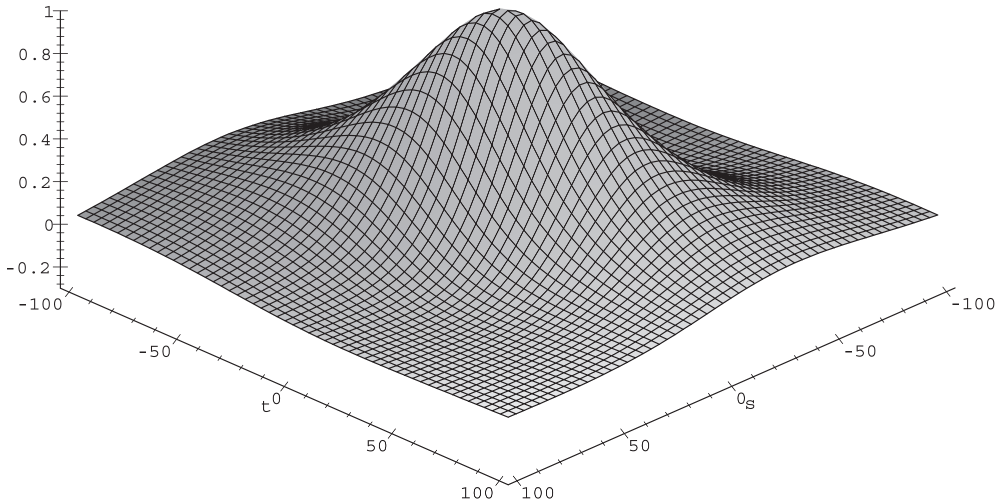

Figure 1. The plot of $C(\frac{\mathbf{s}}{50};\frac{t}{50})/C(0;0)$, where $C(\mathbf{s};t)$ takes the form (3.1), $\alpha_1 = 1$, $\alpha_2 = 2$, $\nu = 1$, $\nu_1 = 2$, $\nu_2 = 1$, $d = 1$ and $\theta_0 = 0.5$.

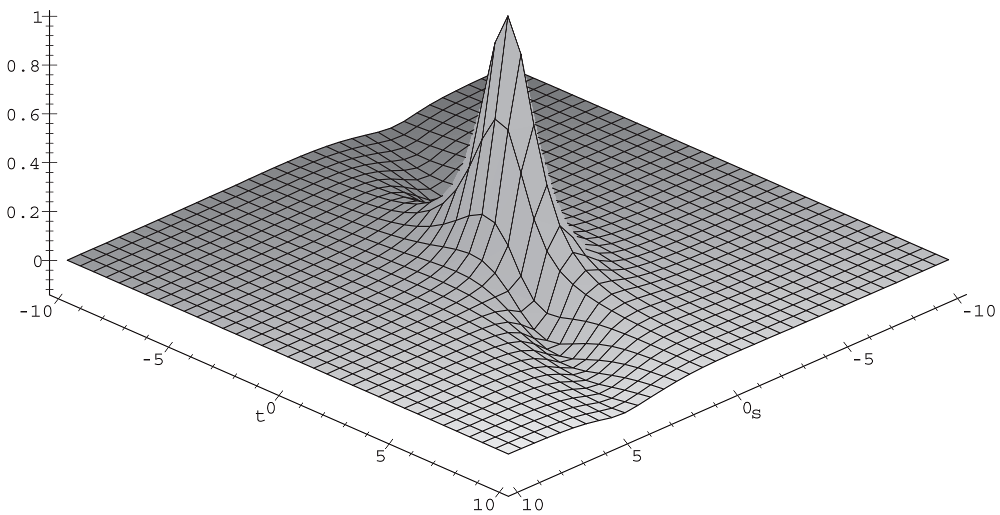

Figure 2. The plot of the correlation function $C(\mathbf{s};t)/C(\mathbf{0};0)$, where $C(\mathbf{s};t)$ is of the form (3.2), $\alpha_1 = 1$, $\alpha_2 = 2$, $b_1 = 3$, $b_2 = 4$, $\nu = 2$, $d = 1$, and $\theta_0 = 0.5$.

$$
\begin{aligned}
C(\mathbf{s};t) = {} & (\|\mathbf{s}-\boldsymbol{\theta}_0 t\|^{\nu}+\alpha_1)^{-\frac{d}{2}} \ln \frac{\|\mathbf{s}+\boldsymbol{\theta}_0 t\|^2 + (\|\mathbf{s}-\boldsymbol{\theta}_0 t\|^{\nu}+\alpha_1) b_2}{\|\mathbf{s}+\boldsymbol{\theta}_0 t\|^2 + (\|\mathbf{s}-\boldsymbol{\theta}_0 t\|^{\nu}+\alpha_1) b_1} \\
& - (\|\mathbf{s}-\boldsymbol{\theta}_0 t\|^{\nu}+\alpha_2)^{-\frac{d}{2}} \ln \frac{\|\mathbf{s}+\boldsymbol{\theta}_0 t\|^2 + (\|\mathbf{s}-\boldsymbol{\theta}_0 t\|^{\nu}+\alpha_2) b_2}{\|\mathbf{s}+\boldsymbol{\theta}_0 t\|^2 + (\|\mathbf{s}-\boldsymbol{\theta}_0 t\|^{\nu}++\alpha_2) b_1}, \qquad (3.2) \\
& \mathbf{s} \in \mathbb{R}^d, \qquad t \in \mathbb{R}.
\end{aligned}
$$

Its smoothness relies on the parameter ν. When $\nu = 2$, (3.2) is differentiable with respect to $\mathbf{s}$ and t. Figure 2 shows a plot of (3.2), which changes sign.

4. How are the parameters involved in the model?

The two parameters α_1 and α_2 are involved in (2.1). It would be helpful to see how the covariance function (2.1) is affected by α_1 and α_2. Corollary 2 may be thought of as an answer to this question, where letting α_2 tend to infinity yields a positive covariance function. Theorem 2 describes the limiting status when α_1 and α_2 are close to each other. Alternatively, one may look at the variance of the associated random field

$$C(\mathbf{s}, \mathbf{s}; t, t) = \alpha_1^{-\frac{d}{2}} - \alpha_2^{-\frac{d}{2}}, \qquad \mathbf{s} \in \mathbb{R}^d, \qquad t \in \mathcal{T}.$$

As d tends to infinity, $C(\mathbf{s}, \mathbf{s}; t, t)$ approaches 0 if $\alpha_2 \geqslant \alpha_1 > 1$, behaves like $\alpha_1^{-\frac{d}{2}}$ if $\alpha_2 > 1 > \alpha_1 > 0$ and like $\alpha_2^{-\frac{d}{2}}$ if $1 \geqslant \alpha_2 > \alpha_1 > 0$.

Next we make a comparison between (2.1) and the same model but with different parameters β_1 and β_2. To this end, let us denote the covariance function (2.1) by $C(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2|\alpha_1, \alpha_2)$ with two parameters α_1 and α_2 involved, and the associated variogram by $\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2|\alpha_1, \alpha_2)$, i.e.

$$\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2|\alpha_1, \alpha_2) = \frac{1}{2}\sum_{k=1}^{2} C(\mathbf{s}_k, \mathbf{s}_k; t_k, t_k|\alpha_1, \alpha_2) - C(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2|\alpha_1, \alpha_2).$$

Similar notations apply to $C(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2|\beta_1, \beta_2)$ and $\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2|\beta_1, \beta_2)$.

Theorem 4. *Let α_k and β_k ($k = 1, 2$) be positive constants with $\alpha_1 < \alpha_2$ and $\beta_1 < \beta_2$. If $\alpha_2 \geqslant \beta_2 > \beta_1 \geqslant \alpha_1$, then*

$$C(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2|\alpha_1, \alpha_2) - C(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2|\beta_1, \beta_2), \qquad (\mathbf{s}_k; t_k) \in \mathbb{R}^d \times \mathcal{T}, \qquad k = 1, 2,$$

is the covariance function of a second-order elliptically contoured random field on $\mathbb{R}^d \times \mathcal{T}$, and

$$\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2|\alpha_1, \alpha_2) - \gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2|\beta_1, \beta_2), \qquad (\mathbf{s}_k; t_k) \in \mathbb{R}^d \times \mathcal{T}, \qquad k = 1, 2,$$

is the variogram of a second-order elliptically contoured random field on $\mathbb{R}^d \times \mathcal{T}$.

Intuitively, theorem 4 claims that when the parameters β_1 and β_2 are between α_1 and α_2, the model with the parameters β_1 and β_2 has a relatively 'small' variation than that with the parameters α_1 and α_2. In the stationary case, it can be shown that the ordinary kriging variance (cf, equation (3.2.17), of [6]) for the model with the parameters β_1 and β_2 is smaller than that with the parameters α_1 and α_2.

A similar comparison can be made between (2.6) and the same model but with different parameters β_1 and β_2.

5. Proofs

Proof of theorem 1. Since $\ell(x)$ is completely monotone on $[0, \infty)$, by Bernstein's theorem, it is the Laplace transform of a bounded, nondecreasing function $F(u)$, $u \geqslant 0$, that is,

$$\ell(x) = \int_0^\infty \exp(-xu)\, \mathrm{d}F(u), \qquad x \geqslant 0.$$

Hence, (2.1) is the same as

$$C(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2) = \{\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2) + \alpha_1\}^{-\frac{d}{2}} \int_0^\infty \exp\left(-\frac{\|\mathbf{s}_2 - \mathbf{s}_1 + \boldsymbol{\theta}_0(t_2 - t_1)\|^2 u}{\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2) + \alpha_1}\right) \mathrm{d}F(u)$$
$$- \{\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2) + \alpha_2\}^{-\frac{d}{2}} \int_0^\infty \exp\left(-\frac{\|\mathbf{s}_2 - \mathbf{s}_1 + \boldsymbol{\theta}_0(t_2 - t_1)\|^2 u}{\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2) + \alpha_2}\right) \mathrm{d}F(u)$$
$$= \int_0^\infty C_u(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2) \mathrm{d}F(u), \qquad \mathbf{s}_1, \mathbf{s}_2 \in \mathbb{R}^d, \qquad t_1, t_2 \in \mathcal{T},$$

where

$$C_u(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2) = \{\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2) + \alpha_1\}^{-\frac{d}{2}} \exp\left(-\frac{\|\mathbf{s}_2 - \mathbf{s}_1 + \boldsymbol{\theta}_0(t_2 - t_1)\|^2 u}{\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2) + \alpha_1}\right)$$
$$- \{\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2) + \alpha_2\}^{-\frac{d}{2}} \exp\left(-\frac{\|\mathbf{s}_2 - \mathbf{s}_1 + \boldsymbol{\theta}_0(t_2 - t_1)\|^2 u}{\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2) + \alpha_2}\right),$$

$u \geqslant 0, \quad \mathbf{s}_1, \mathbf{s}_2 \in \mathbb{R}^d, \quad t_1, t_2 \in \mathcal{T}.$

Therefore, it suffices to show that for every constant $u \geqslant 0$, $C_u(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2)$ is a covariance function on $\mathbb{R}^d \times \mathcal{T}$.

Using the formula

$$(2\pi)^{d/2}|A|^{-1/2} \exp\left(-\frac{1}{2}\mathbf{x}' A^{-1}\mathbf{x}\right) = \int_{\mathbb{R}^d} \exp\left(\imath \boldsymbol{\omega}'\mathbf{x} - \frac{1}{2}\boldsymbol{\omega}' A \boldsymbol{\omega}\right) \mathrm{d}\boldsymbol{\omega}, \qquad \mathbf{x} \in \mathbb{R}^d,$$

and substituting $\mathbf{x}$ by $\{\mathbf{s}_2 - \mathbf{s}_1 + \boldsymbol{\theta}_0(t_2 - t_1)\}\sqrt{2u}$ and A by $\{\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2) + \alpha_1\}I_d$ or $\{\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2) + \alpha_2\}I_d$, where I_d is a $d \times d$ identity matrix, we rewrite $C_u(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2)$ as

$$C_u(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \cos\{\sqrt{2u}\boldsymbol{\omega}'(\mathbf{s}_2 - \mathbf{s}_1 + \boldsymbol{\theta}_0(t_2 - t_1))\}$$
$$\times \exp\left\{-\frac{1}{2}\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2)\|\boldsymbol{\omega}\|^2\right\} \left\{\exp\left(-\frac{\alpha_1}{2}\|\boldsymbol{\omega}\|^2\right) - \exp\left(-\frac{\alpha_2}{2}\|\boldsymbol{\omega}\|^2\right)\right\} \mathrm{d}\boldsymbol{\omega}.$$

This is indeed a covariance function on $\mathbb{R}^d \times \mathcal{T}$ as a mixture of the nonnegative function $\exp\left(-\frac{\alpha_1}{2}\|\boldsymbol{\omega}\|^2\right) - \exp\left(-\frac{\alpha_2}{2}\|\boldsymbol{\omega}\|^2\right)$ on $\mathbb{R}^d$, because for every $\boldsymbol{\omega} \in \mathbb{R}^d$, $\cos\left\{\sqrt{2u}\boldsymbol{\omega}'(\mathbf{s}_2 - \mathbf{s}_1 + \boldsymbol{\theta}_0(t_2 - t_1))\right\}$ is a covariance function on $\mathbb{R}^d \times \mathcal{T}$, and by Schoenberg's theorem, $\exp\left\{-\frac{1}{2}\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2)\|\boldsymbol{\omega}\|^2\right\}$ is a covariance function on $\mathbb{R}^d \times \mathcal{T}$. □

Proof of theorem 2. Let us introduce an auxiliary variable δ on the interval $(0, 1)$, and consider the function

$$C(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2; \delta) = \{\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2) + \alpha + \delta\}^{-\frac{d}{2}} \ell\left(\frac{\|\mathbf{s}_2 - \mathbf{s}_1 + \boldsymbol{\theta}_0(t_2 - t_1)\|^2}{\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2) + \alpha + \delta}\right),$$

$\mathbf{s}_1, \mathbf{s}_2 \in \mathbb{R}^d, \qquad t_1, t_2 \in \mathcal{T}.$

According to theorem 1, $C(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2; 0) - C(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2; \delta)$ is a covariance function on $\mathbb{R}^d \times \mathcal{T}$, so is

$$\delta^{-1}\{C(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2; 0) - C(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2; \delta)\}, \qquad \mathbf{s}_1, \mathbf{s}_2 \in \mathbb{R}^d, \qquad t_1, t_2 \in \mathcal{T}.$$

Letting δ approach 0 yields a new covariance function on $\mathbb{R}^d \times \mathcal{T}$, which is, by L'Hospital's rule,

$$\lim_{\delta \to 0_+} \delta^{-1}\{C(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2; 0) - C(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2; \delta)\} = -\lim_{\delta \to 0_+} \frac{\partial}{\partial \delta} C(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2; \delta)$$
$$= C(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2),$$

and thus coincides with (2.5). □

Proof of theorem 3. The proofs of parts (ii) and (iii) are similar to those of corollary 3. To show part (i), note that, by Schoenberg's theorem, $\exp\{-\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2)u\}$ is a covariance function on $\mathbb{R}^d \times \mathcal{T}$ for every fixed $u \geqslant 0$, so is its nonnegative mixture

$$\frac{1}{\Gamma(\nu)} \int_0^\infty \exp\{-\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2)u\}\{\exp(-\alpha_1 u) - \exp(-\alpha_2 u)\}u^{\nu-1}\mathrm{d}u$$
$$= \{\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2) + \alpha_1\}^{-\nu} - \{\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2) + \alpha_2\}^{-\nu}. \qquad \square$$

Proof of theorem 4. Using the approach in the proof of theorem 1, we rewrite $C(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2 |\alpha_1, \alpha_2) - C(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2|\beta_1, \beta_2)$ as

$$C(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2|\alpha_1, \alpha_2) - C(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2|\beta_1, \beta_2)$$
$$= \int_0^\infty \{C_u(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2|\alpha_1, \alpha_2) - C_u(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2|\beta_1, \beta_2)\}\,\mathrm{d}F(u),$$
$$\mathbf{s}_1, \mathbf{s}_2 \in \mathbb{R}^d, \qquad t_1, t_2 \in \mathcal{T},$$

where

$$C_u(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2|\alpha_1, \alpha_2) = \{\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2) + \alpha_1\}^{-\frac{d}{2}} \exp\left(-\frac{\|\mathbf{s}_2 - \mathbf{s}_1 + \boldsymbol{\theta}_0(t_2 - t_1)\|^2 u}{\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2) + \alpha_1}\right)$$
$$- \{\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2) + \alpha_2\}^{-\frac{d}{2}} \exp\left(-\frac{\|\mathbf{s}_2 - \mathbf{s}_1 + \boldsymbol{\theta}_0(t_2 - t_1)\|^2 u}{\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2) + \alpha_2}\right),$$

and

$$C_u(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2|\beta_1, \beta_2) = \{\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2) + \beta_1\}^{-\frac{d}{2}} \exp\left(-\frac{\|\mathbf{s}_2 - \mathbf{s}_1 + \boldsymbol{\theta}_0(t_2 - t_1)\|^2 u}{\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2) + \beta_1}\right)$$
$$- \{\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2) + \beta_2\}^{-\frac{d}{2}} \exp\left(-\frac{\|\mathbf{s}_2 - \mathbf{s}_1 + \boldsymbol{\theta}_0(t_2 - t_1)\|^2 u}{\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2) + \beta_2}\right),$$
$$u \geqslant 0, \qquad \mathbf{s}_1, \mathbf{s}_2 \in \mathbb{R}^d, \qquad t_1, t_2 \in \mathcal{T}.$$

Also, $C_u(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2|\alpha_1, \alpha_2) - C_u(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2|\beta_1, \beta_2)$ can be expressed as

$$C_u(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2|\alpha_1, \alpha_2) - C_u(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2|\beta_1, \beta_2)$$
$$= (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \cos\{\sqrt{2u}\boldsymbol{\omega}'(\mathbf{s}_2 - \mathbf{s}_1 + \boldsymbol{\theta}_0(t_2 - t_1))\} \exp\left\{-\frac{1}{2}\gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2)\|\boldsymbol{\omega}\|^2\right\}$$
$$\times \left\{\exp\left(-\frac{\alpha_1}{2}\|\boldsymbol{\omega}\|^2\right) + \exp\left(-\frac{\beta_2}{2}\|\boldsymbol{\omega}\|^2\right)\right.$$
$$\left. - \exp\left(-\frac{\alpha_2}{2}\|\boldsymbol{\omega}\|^2\right) - \exp\left(-\frac{\beta_1}{2}\|\boldsymbol{\omega}\|^2\right)\right\} \mathrm{d}\boldsymbol{\omega},$$

which is a covariance function on $\mathbb{R}^d \times \mathcal{T}$ once we are able to verify that

$$\exp\left(-\frac{\alpha_1}{2}\|\boldsymbol{\omega}\|^2\right) + \exp\left(-\frac{\beta_2}{2}\|\boldsymbol{\omega}\|^2\right) - \exp\left(-\frac{\alpha_2}{2}\|\boldsymbol{\omega}\|^2\right)$$
$$- \exp\left(-\frac{\beta_1}{2}\|\boldsymbol{\omega}\|^2\right) \geqslant 0, \qquad \boldsymbol{\omega} \in \mathbb{R}^d. \tag{5.1}$$

In fact, since $\exp(-x\|\boldsymbol{\omega}\|)$, $x \geqslant 0$, is a convex and decreasing function of $x \in [0, \infty)$, the function

$$\phi(x_1, x_2) = \exp(-x_1\|\boldsymbol{\omega}\|) + \exp(-x_2\|\boldsymbol{\omega}\|), \qquad x_1, x_2 \geqslant 0,$$

is Schur-convex on $[0, \infty) \times [0, \infty)$ (see proposition C.1, p 64, of [28]), for a fixed $\boldsymbol{\omega} \in \mathbb{R}^d$. By assumption, $\alpha_1 < \alpha_2$, $\alpha_1 \leqslant \beta_1 < \beta_2$, and $\alpha_1 + \beta_2 \leqslant \beta_1 + \alpha_2$, so that the vector $(\alpha_2/2, \beta_1/2)$

is weakly supermajorized by the vector $(\alpha_1/2, \beta_2/2)$. Thus, it follows from proposition C.1.b., p 64, of [28] that

$$\phi\left(\frac{\alpha_2}{2}, \frac{\beta_1}{2}\right) \leqslant \phi\left(\frac{\alpha_1}{2}, \frac{\beta_2}{2}\right),$$

which confirms inequality (5.1).

Finally, the variogram associated with $C(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2\,|\alpha_1, \alpha_2) - C(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2|\beta_1, \beta_2)$ is

$$\begin{aligned}
&\frac{1}{2}\sum_{k=1}^{2}\{C(\mathbf{s}_k, \mathbf{s}_k; t_k, t_k|\alpha_1, \alpha_2) - C(\mathbf{s}_k, \mathbf{s}_k; t_k, t_k|\beta_1, \beta_2)\}\\
&\qquad\qquad - \{C(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2|\alpha_1, \alpha_2) - C(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2|\beta_1, \beta_2)\}\\
&\qquad = \gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2|\alpha_1, \alpha_2) - \gamma(\mathbf{s}_1, \mathbf{s}_2; t_1, t_2|\beta_1, \beta_2).
\end{aligned}$$

□

6. Discussions

To have a better description of a random field, it is necessary to know its finite-dimensional distributions. In the Gaussian case, its finite-dimensional distributions are completely specified by its mean and covariance functions, and, more importantly, there is not a restriction or a tight connection between its mean and covariance functions, unlike log-Gaussian or χ^2 cases [27], [29] so that the Gaussian random field can be relatively easily handled for applications. The class of elliptically contoured random fields contains the Gaussian case as a particular case, and keeps some important Gaussian features. For instance, finite-dimensional distributions of a second-order elliptically contoured random field are completely specified by its mean function and its covariance function that is fully characterized by the positive definiteness. The advantage of being easy to manipulate analytically would make second-order elliptically contoured random fields work effectively for studying various correlation effects in physics and applied sciences.

The main result of this paper is the derivation of a class of the spacetime Gaussian and second-order elliptically contoured random fields having a nonseparable covariance function by using the nonnegative mixture approach. The covariance functions have been shown to allow both positive and negative correlations. Some properties of these random fields, in particular, their spacetime stationarity, long-range dependence, parameter dependence, have been studied. Below we give a couple of examples showing the potential use of this formulation in some specific physical situation where non-Gaussian statistics is demanded.

Liebovitch *et al* [20] studied physiological relevance of scaling of heart phenomena, with heart rate data from people who are healthy, from people who have a specific sleep disorder and from people who have irregular heart rhythms, and found that the data have fractal rather than Gaussian distributions or have long-range correlation. A further investigation of their experimental data may be performed for what they called the power-law form of the PDF, or Student's t random field [26], which is an elliptically contoured random field. Other examples of the power-law form of the PDF may be found in [4], [5], [11].

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References

[1] Adler R J 1990 The net charge process for interacting, signed diffusions *Ann. Prob.* **18** 602–25
[2] Armstrong M 1992 Positive definiteness is not enough *Math. Geol.* **24** 135–43
[3] Baeumer B, Benson D A and Meerschaert M M 2005 Advection and dispersion in time and space *Physica* A **350** 245–62
[4] Borgas M S and Yeung P K 2004 Relative dispersion in isotropic turbulence. Part 2. A new stochastic model with Reynolds-number dependence *J. Fluid Mech.* **503** 125–60
[5] Bray A J and Majumdar S N 2006 Partial survival and crossing statistics for a diffusing particle in a transverse shear flow *J. Phys. A: Math. Gen.* **39** L625–31
[6] Cressie N 1993 *Statistics for Spatial Data* revised edn (New York: Wiley)
[7] Cressie N and Huang H C 1999 Classes of nonseparable, spatio-temporal stationary covariance functions *J. Am. Stat. Assoc.* **94** 1330–40
[8] Deloubriére O and Hilhorst H J 2000 Persistence exponents of non-Gaussian processes in statistical mechanics *J. Phys. A: Math. Gen.* **33** 1993–2013
[9] Doukhan P, Oppenheim G and Taqqu M S (eds) 2003 *Theory and Applications of Long-range Dependence* (Boston, MA: Birkhauser)
[10] Esary J D, Proschan F and Walkup D W 1967 Association of random variables, with applications *Ann. Math. Stat.* **38** 1466–74
[11] Fraedrich K and Larnder C 1993 Scaling regimes of composite rainfall time series *Tellus* A **45** 289–98
[12] Giacomini R and Granger C W J 2004 Aggregation of space-time processes *J. Econometrics* **118** 7–26
[13] Gioffré M and Gusella V 2002 Numerical analysis of structural systems subjected to non-Gaussian random fields *Meccanica* **37** 115–28
[14] Grigoriu M 1995 *Applied Non-Gaussian Processes: Examples, Theory, Simulation, Linear Random Vibration, and MATLAB Solutions* (Englewood Cliffs, NJ: Prentice-Hall)
[15] Holmes J D 1981 Non-Gaussian characteristics of wind pressure fluctuations *J. Wind Eng. Ind. Aerodyn.* **7** 103–8
[16] Jensen D R and Foutz R V 1989 The structure and analysis of spherical time-dependent processes *SIAM J. Appl. Math.* **49** 1834–44
[17] Joag-Dev K, Perlman M D and Pitt L D 1983 Association of normal random variables and Slepian's inequality *Ann. Prob.* **11** 451–5
[18] Jones R H and Zhang Y 1997 Models for continuous stationary space-time processes *Modelling Longitudinal and Spatially Correlated Data* (*Lecture Notes in Statistics* vol 122) ed T G Gregoire *et al* (Berlin: Springer) pp 289–98
[19] Le N D and Zidek J V 2006 *Statistical Analysis of Environmental Space-time Processes* (New York: Springer)
[20] Liebovitch L S, Penzel T and Kantelhardt J W 2002 Physiological relevance of scaling of heart phenomena *The Science of Disasters: Climate Disruptions, Heart Attacks, and Market Crashes* ed A Bunde, J Kropp and H J Schellnhuber (Berlin: Springer) pp 259–82
[21] Lim S C and Teo L P 2009 Generalized Whittle-Matérn random field as a model of correlated fluctuations *J. Phys. A: Math. Theor.* **42** 105202
[22] Ma C 2003 Nonstationary covariance functions that model space-time interactions *Stat. Prob. Lett.* **62** 411–9
[23] Ma C 2005 Linear combinations of space-time covariance functions and variograms *IEEE Trans. Signal Process.* **53** 857–64
[24] Ma C 2005 Spatio-temporal variograms and covariance models *Adv. Appl. Prob.* **37** 706–25
[25] Ma C 2007 Stationary random fields in space and time with rational spectral densities *IEEE Trans. Inf. Theory* **53** 1019–29
[26] Ma C 2009 Construction of non-Gaussian random fields with any given correlation structure *J. Stat. Plan. Inference* **139** 780–7
[27] Ma C 2010 χ^2 random fields in space and time *IEEE Trans. Signal Process.* **58** 378–83
[28] Marshall A W and Olkin I 1979 *Inequalities: Theory of Majorization and Its Applications* (New York: Academic)
[29] Matheron G 1989 The internal consistency of models in geostatistics *Geostatistics* vol 1 ed M Armstrong (Dordrecht: Kluwer) pp 21–38
[30] Miller K S and Samko S G 2001 Completely monotonic functions *Integral Transform. Spec. Funct.* **12** 389–402
[31] Newsam G N and Wegener M 1994 Generating non-Gaussian random fields for sea surface simulations *ICASSP-94: Int. Conf. on Acoustics, Speech, and Signal Processing* (Adelaide, 1994) vol 6 pp 195–8
[32] Pearce I G, Chaplain M A J, Schofield P G, Anderson A R A and Hubbard S F 2006 Modelling the spatio-temporal dynamics of multi-species host-parasitoid interactions: heterogeneous patterns and ecological implications *J. Theor. Biol.* **241** 876–86

[33] Pereira G M 2002 A typology of spatial and temporal scale relation *Geographical Anal.* **34** 21–33
[34] Phillips W R C 2000 Eulerian space-time correlations in turbulent shear flows *Phys. Fluids* **12** 2056–64
[35] Pitt L D 1982 Positively correlated normal variables are associated *Ann. Prob.* **10** 496–9
[36] Rao P S, Johnson D H and Becker D D 1992 Generation and analysis of non-Gaussian Markov time series *IEEE Trans. Signal Process.* **40** 845–56
[37] Rodriguez Iturbe I, Marani M, D'Odorico P and Rinaldo A 1998 On space-time scaling of cumulated rainfall fields *Water Resources Res.* **34** 3461–9
[38] Stefanou G 2009 The stochastic finite element method: past, present and future *Comput. Methods Appl. Mech. Eng.* **198** 1031–51
[39] Tong Y L 1990 *The Multivariate Normal Distribution* (New York: Springer)
[40] Weber R O and Talkner P 1993 Some remarks on spatial correlation function models *Mon. Weather Rev.* **121** 2611–7
[41] Vershik A M 1964 Some characteristic properties of stochastic Gaussian processes *Theory Prob. Appl.* **9** 353–6